Fundamental Theorem of Calculus

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Abstract

The fundamental theorem of calculus is a theorem that links the concept of differentiating a function with the concept of integrating a function. In this note we discuss the fundamental theorem of calculus and its proof.

Introduction

The fundamental theorem of calculus relates differentiation and integration, showing that these two operations are essentially inverses of one another. "This result was known to NEWTON (1642–1727) and even—in a geometric form—to NEWTON's teacher BARROW (1630–1677), but it became more transparent in LEIBNIZ's formalism". [1]

1 The Theorem and Its Proof

Let f be a continuous function on an open interval $I \subseteq \mathbf{R}$. Fix a point $a \in I$. For any point $x \in I$, we define

$$F(x) = \int_a^x f(t)dt \tag{\dagger}$$

The substance of the *Fundamental Theorem of Calculus* is to claim that the function F is an anti-derivative for f. More precisely, we have

Theorem 1. *The function F defined above is differentiable, and*

$$\frac{d}{dx}F(x) = f(x)$$

for every $x \in I$.

Proof. We endeavor to calculate the derivative of F by forming the difference or Newton quotient for $h \neq 0$:

$$\begin{aligned} \underbrace{\frac{F(x+h) - F(x)}{h}}_{Newton\ qoutient} &= \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} \\ &= \frac{\int_x^{x+h} f(t)dt}{h}. \end{aligned} \tag{1}$$

Now fix a point $x \in I$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $|t - x| < \delta$ implies that $|f(t) - f(x)| < \epsilon$. Now we may rewrite (1) as

$$\frac{\int_x^{x+h} f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h} + \frac{\int_x^{x+h}[f(t) - f(x)]dt}{h}$$
$$= f(x) + \frac{\int_x^{x+h}[f(t) - f(x)]dt}{h}.$$

If $|h| < \delta$, then we may estimate the last fraction as

$$\left|\frac{\int_x^{x+h}[f(t) - f(x)]dt}{h}\right| \leq \frac{\int_x^{x+h} |f(t) - f(x)|dt}{h} \leq \epsilon.$$

Thus, in summary, we have

$$\frac{F(x+h) - F(x)}{h} = f(x) + error,$$

where the error is not greater than ϵ. In conclusion,

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

as desired.

□

References

[1] Stillwell, John. *Mathematics and its History* (Undergraduate Texts in Mathematics). 3rd ed. Springer Science & Business Media, 2010.